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NOTE ON THE ROOTS OF BESSEL FUNCTIONS

BY CHARLES N. MOORE

THE earliest proof of the fact that the Bessel function of order zero, $J_0(x)$, has an infinite number of roots was given by Bessel in the year 1824* and was obtained directly from the integral definition of $J_0(x)$:

$$(1) \qquad J_0(x) = \frac{2}{\pi}\int_0^1 \frac{\cos xt}{\sqrt{1-t^2}}\,dt.$$

In the course of his proof Bessel showed that $J_0(x)$ has at least one root in each of the intervals

$$(2) \qquad (m+\tfrac{1}{2})\pi < x < (m+1)\pi \qquad (m = 0, 1, 2, \cdots)$$

and no positive roots outside of these intervals.

The object of this paper is to establish the fact that $J_0(x)$ has only one root in each of the intervals (2); this will be shown by applying methods analogous to those of Bessel to the integral definition of $J_1(x)$:†

$$(3) \qquad J_1(x) = \frac{2x}{\pi}\int_0^1 \sqrt{1-t^2}\cos xt\,dt.$$

Although the result itself is not new it seems worth while, from a historical point of view at least, to give a proof of it which is analogous to Bessel's proof of the less specific fact.

The outline of the method is as follows: It is a well known fact that

$$\frac{dJ_0(x)}{dx} = -J_1(x).\ddagger$$

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* Cf. *Berliner Abhandlungen* (1824), *Mathematische Klasse*, p. 39. For a reproduction of Bessel's proof see Gray and Mathews, *Treatise on Bessel Functions*, p. 44.

† This fact can also be obtained directly from the expression (1) but the proof in that case is somewhat more complicated.

‡ Cf. Gray and Mathews, *Treatise on Bessel Functions*, p. 13.

We shall prove that

$$\operatorname{sgn} J_1(x) = (-1)^m \qquad (m + \tfrac{1}{2})\pi \leqq x \leqq (m + 1)\pi,$$

and hence that for this interval

$$\operatorname{sgn} \frac{dJ_0(x)}{dx} = (-1)^{m+1}.$$

Then according as m is odd or even $J_0(x)$ continually increases or continually decreases throughout the interval, and therefore in either case it cannot have more than one root in that interval. Combining this with the fact proved by Bessel, namely that $J_0(x)$ has at least one root in the interval, it follows that it has just one root in every such interval.

We consider values of x that lie in the interval

$$(m + \tfrac{1}{2})\pi \leqq x \leqq (m + 1)\pi$$

and we set

$$x = \frac{2m + 1 + m'}{2}\pi \qquad (0 \leqq m' \leqq 1). \tag{4}$$

If now we make the change of variable

$$v = (2m + 1 + m')t$$

and for brevity put

$$\mu = 2m + 1 + m', \tag{5}$$

equation (3) reduces with the help of (4) to

$$J_1(x) = \frac{2x}{\pi}\int_0^{2m+1+m} \frac{1}{\mu^2}\cos\frac{\pi v}{2}\sqrt{\mu^2 - v^2}\,dv. \tag{6}$$

If then we set

$$u_n = \int_{2n}^{2n+2} \cos\frac{\pi v}{2}\sqrt{\mu^2 - v^2}\,dv \qquad (n = 0, 1, 2, \cdots (m - 1)), \tag{7}$$

and

$$I = \int_{2m}^{2m+1+m'} \cos\frac{\pi v}{2}\sqrt{\mu^2 - v^2}\,dv, \tag{8}$$

we may write (6) in the form

$$J_1(x) = \frac{2x}{\pi\mu^2}\left[\sum_{n=0}^{n=m-1} u_n + I\right]. \tag{9}$$

Since the quantity $\sqrt{\mu^2 - v^2}$, which appears as a factor in the integrand of the integral on the right-hand side of (7), continually decreases with increasing v as long as v lies in the interval $0 \leqq v \leqq \mu$ it follows that this quantity is less in absolute value in the second half of the interval of integration of the integral in question than it is in the first half. Hence the integral has the same sign that the integrand has in the first half of the interval, i. e.,

$$\operatorname{sgn} u_n = \operatorname{sgn} \cos n\pi = (-1)^n.$$

We see thus that the finite sum

$$\sum_{n=0}^{n=m-1} u_n$$

has its terms alternating in sign. We will show next that these terms are continually increasing in absolute value.

If we set

$$v' = v + 2,$$

we get for u_{n-1}

$$u_{n-1} = -\int_{2n}^{2n+2} \cos\frac{\pi v'}{2}\sqrt{\mu^2 - (v'-2)^2}\, dv'.$$

Consequently

$$u_n + u_{n-1} = \int_{2n}^{2n+2} \cos\frac{\pi v}{2}\left[\sqrt{\mu^2 - v^2} - \sqrt{\mu^2 - (v-2)^2}\right] dv.$$

The quantity in brackets in the integrand of the last integral is obviously negative throughout the interval of integration; moreover it decreases with increasing v since its derivative with regard to v is

$$-\frac{v}{\sqrt{\mu^2 - v^2}} + \frac{v-2}{\sqrt{\mu^2 - (v-2)^2}},$$

which is a negative quantity. It therefore increases in absolute value with increasing v and hence the integral in question has the sign of the integrand in the second half of the interval, i. e.,

$$\operatorname{sgn}[u_n + u_{n-1}] = -\operatorname{sgn}\cos(n+1)\pi = (-1)^n = \operatorname{sgn} u_n,$$

and this shows, since u_n and u_{n-1} have opposite signs, that u_n is greater in absolute value than u_{n-1}.

We now wish to show that the quantity I, defined by (8), has the sign $(-1)^m$ and is greater in absolute value than u_{m-1}. By breaking up the integral in (8) we get

$$(10)\qquad I = \int_{2m}^{2m+1-m'} \cos\frac{\pi v}{2}\sqrt{\mu^2 - v^2}\,dv + \int_{2m+1-m'}^{2m+1+m'} \cos\frac{\pi v}{2}\sqrt{\mu^2 - v^2}\,dv.$$

The first integral on the right-hand side of (10) has the sign $(-1)^m$ since its integrand has that sign throughout the interval of integration; the second integral has the sign $(-1)^m$ since its integrand has that sign in the first half of the interval of integration and the quantity $\sqrt{\mu^2 - v^2}$ is greater in absolute value there than it is in the second half of the interval of integration. Hence I has the sign $(-1)^m$. If we can show further that the sum of I and u_{m-1} has the sign $(-1)^m$ it will follow that I is greater in absolute value than u_{m-1}, since the latter has the sign $(-1)^{m-1}$.

When we make the change of variable

$$v = v' + 2,$$

we get for I

$$I = -\int_{2m-2}^{2m-1+m'} \cos\frac{\pi v'}{2}\sqrt{\mu^2 - (v'+2)^2}\,dv',$$

so that we have

$$(11)\qquad u_{m-1} + I = \int_{2m-2}^{2m-1+m'} \cos\frac{\pi v}{2}\left[\sqrt{\mu^2 - v^2} - \sqrt{\mu^2 - (v+2)^2}\right]dv$$

$$+ \int_{2m-1+m'}^{2m} \cos\frac{\pi v}{2}\sqrt{\mu^2 - v^2}\,dv.$$

If, in the last integral, we set

$$v = 4m - 2 - v'$$

it becomes

$$-\int_{2m-2}^{2m-1-m'} \cos\frac{\pi v'}{2}\sqrt{\mu^2 - (4m - 2 - v')^2}\,dv',$$

so that we may write (11) in the form

$$(12)\qquad u_{m-1} + I$$

$$= \int_{2m-2}^{2m-1-m'} \cos\frac{\pi v}{2}\left[\sqrt{\mu^2 - v^2} - \sqrt{\mu^2 - (v+2)^2} - \sqrt{\mu^2 - (4m-2-v)^2}\right] dv$$

$$+ \int_{2m-1-m'}^{2m-1+m'} \cos\frac{\pi v}{2}\left[\sqrt{\mu^2 - v^2} - \sqrt{\mu^2 - (v+2)^2}\right] dv.$$

The expression in brackets in the integrand of the second integral on the right-hand side of (12) is obviously positive; moreover it increases with increasing v since its derivative

$$-\frac{v}{\sqrt{\mu^2 - v^2}} + \frac{v+2}{\sqrt{\mu^2 - (v+2)^2}}$$

is a positive quantity. Hence the integral in question has the sign of the integrand in the second half of the interval of integration, that is $(-1)^m$.

Let us write

$$(13)\qquad \psi(v) = \sqrt{\mu^2 - v^2} - \sqrt{\mu^2 - (v+2)^2} - \sqrt{\mu^2 - (4m-2-v)^2}.$$

On substituting the value $(2m - 2)$ for v we get

$$(14)\qquad \psi(2m-2) = \sqrt{\mu^2 - (2m-2)^2} - 2\sqrt{\mu^2 - 4m^2}.$$

Since in view of (5) μ is greater than or equal to $2m + 1$, we have

$$3\mu^2 \geqq 12m^2 + 12m + 3 > 12m^2 + 8m - 4,$$

or, adding $(\mu^2 - 16m^2)$ to both sides of the inequality,

$$4\mu^2 - 16m^2 > \mu^2 - 4m^2 + 8m - 4 = \mu^2 - (2m-2)^2.$$

This gives, on extracting square roots,

$$2\sqrt{\mu^2 - 4m^2} > \sqrt{\mu^2 - (2m-2)^2},$$

which, in combination with (14), shows that $\psi(2m-2)$ is a negative quantity.

Furthermore $\psi(2m-1)$ is either negative or zero, for we have from (13)

$$\begin{aligned}\psi(2m-1) &= \sqrt{\mu^2-(2m-1)^2}-\sqrt{\mu^2-(2m+1)^2}-\sqrt{\mu^2-(2m-1)^2}\\ &= -\sqrt{\mu^2-(2m+1)^2} \leqq 0.\end{aligned}$$

In order to determine the sign of $\psi(v)$ for values of v lying between $(2m-2)$ and $(2m-1)$ we examine its second derivative. We have

$$(15)\qquad \psi''(v) = -\frac{1}{\sqrt{\mu^2-v^2}}-\frac{v^2}{(\mu^2-v^2)^{\frac{3}{2}}}+\frac{1}{\sqrt{\mu^2-(v+2)^2}}$$
$$+\frac{(v+2)^2}{(\mu^2-(v+2)^2)^{\frac{3}{2}}}+\frac{1}{\sqrt{\mu^2-(4m-2-v)^2}}+\frac{(4m-2-v)^2}{(\mu^2-(4m-2-v)^2)^{\frac{3}{2}}}.$$

The function $\psi''(v)$ is obviously positive for all values of v lying in the interval

$$(16)\qquad 2m-2 \leqq v \leqq 2m-1$$

since the first term on the right-hand side of (15) is less in absolute value than the third and the second less in absolute value than the fourth throughout that interval. Hence $\psi'(v)$ is continually increasing throughout the interval (16).

We have then three possibilities for $\psi'(v)$. It may be negative throughout the interval (16), or it may be first negative and then positive, or it may be positive throughout the interval. In none of these cases can $\psi(v)$, which is negative at the beginning of the interval (16) and negative or zero at the end of it, become positive for any value of v lying in the interval. Hence we have

$$\operatorname{sgn}\psi(v) = -1 \qquad (2m-2 \leqq v \leqq 2m-1).$$

It follows, then, that the integrand of the first integral on the right-hand side of (12) has the sign $(-1)^m$ throughout the interval of integration, and therefore the integral has that sign. The whole right-hand side of (12) has then the sign $(-1)^m$. Consequently the left-hand side has that sign also and hence

$$|I| \geqq |u_{m-1}|.$$

Thus we see that the set of quantities

$$u_1, u_2, \cdots u_{m-1}, I$$

alternate in sign and continually increase in absolute value. Their sum, therefore, has the sign of I, for according as the number of terms is odd or even we can write that sum

$$u_1 + (u_2 + u_3) + \cdots + (u_{m-1} + I)$$

or

$$(u_1 + u_2) + (u_3 + u_4) + \cdots + (u_{m-1} + I).$$

We have then, in view of (9),

$$\operatorname{sgn} J_1(x) = (-1)^m \qquad (m + \tfrac{1}{2})\pi \leqq x \leqq (m + 1)\pi,$$

and hence, as we have pointed out above, our theorem is proved.

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